Problem 19

Evaluate $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)$.

Solution

Recall how the integral is defined.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\,\Delta x,$$

where

$$x_i = a + i\,\Delta x \quad \text{and} \quad \Delta x = \frac{b-a}{n}.$$

The strategy for this problem is to write the sum compactly with an index, i, and to change the limit to an integral we can solve. We can write the given expression as follows.

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right) = \lim_{n\to\infty}\sum_{i=1}^n \frac{1}{\sqrt{n}\sqrt{n+i}}$$

Rewrite it similarly to the definition.

$$\lim_{n\to\infty}\sum_{i=1}^n \frac{1}{\sqrt{n+i}}\cdot\frac{1}{\sqrt{n}}$$

The term containing i is going to be $f(x_i)$, and the term that doesn't have i will be Δx. According to the definition, Δx has n in the denominator, not $\sqrt{n}$, so we'll multiply the numerator and denominator by $\sqrt{n}$ to make them the same.

$$\lim_{n\to\infty}\sum_{i=1}^n \frac{1}{\sqrt{n+i}}\cdot\frac{1}{\sqrt{n}}\cdot\frac{\sqrt{n}}{\sqrt{n}} = \lim_{n\to\infty}\sum_{i=1}^n \frac{\sqrt{n}}{\sqrt{n+i}}\cdot\frac{1}{n}$$

Combine the square roots into one and then multiply the resulting fraction's numerator and denominator by $1/n$.

$$\lim_{n\to\infty}\sum_{i=1}^n \sqrt{\frac{n}{n+i}\cdot\frac{1/n}{1/n}}\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^n \sqrt{\frac{1}{1+\frac{i}{n}}}\cdot\frac{1}{n}$$

Comparing the two terms in the sum to the definition, we see that

$$f(x_i) = \sqrt{\frac{1}{1+\frac{i}{n}}} \quad \text{and} \quad \Delta x = \frac{1}{n}$$

and that $a = 1$ and $b - a = 1$, which means $b = 2$. Now that we know the function and the limits of integration, we can write the limit as an integral and solve it.

$$\lim_{n\to\infty}\sum_{i=1}^n \sqrt{\frac{1}{1+\frac{i}{n}}}\cdot\frac{1}{n} = \int_1^2\sqrt{\frac{1}{x}}\,dx = \int_1^2 x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big|_1^2 = 2(\sqrt{2}-\sqrt{1}) = 2(\sqrt{2}-1)$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right) = 2(\sqrt{2}-1).$$